

A gp action $G \curvearrowright S$

is a map $G \times S \rightarrow S$
 $(g, s) \mapsto g \cdot s$

satisfying $1 \cdot s = s \quad \forall s \in S$

$g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s \quad \forall s \in S, \forall g_1, g_2 \in G.$

$\Rightarrow m_g: S \rightarrow S$ is a bijection.
 $s \mapsto g \cdot s$

Defⁿ If $s \in S$, then the orbit O_s of s is

$$O_s = \{s' \in S : \exists g \in G \text{ so that } s' = g \cdot s\}$$

The orbits are equivalence classes for the equiv. relⁿ

$$s \sim s' \text{ if } \exists g \in G \text{ so that } s' = g \cdot s.$$

$\Rightarrow S$ is partitioned into the orbits for the gp action

$$\text{e.g. } S = O_{s_1} \sqcup O_{s_2} \sqcup O_{s_3} \quad (\text{disjoint union})$$

Defⁿ A gp action $G \curvearrowright S$ is said to be transitive

if there is one orbit. Equivalently, given $s_1, s_2 \in S \exists g \in G$ so that $s_2 = g \cdot s_1$.

Defⁿ $G \curvearrowright S$, $s \in S$. Then the stabilizer of s is

$$G_s = \{g \in G : gs = s\}.$$

Claim: G_s is a subgroup of G .

Pf: • $1 \in G_s$ b/c $1 \cdot s = s$

• If $gs = s$ then applying g^{-1} gives $g^{-1}(gs) = g^{-1}(s)$

$$\Rightarrow g^{-1}s = g^{-1}(gs) = (g^{-1}g)s = 1 \cdot s = s$$

• If $g_1s = s$ and $g_2s = s$ then

$$(g_1g_2)s = g_1(g_2s) = g_1(s) = s \Rightarrow g_1g_2 \in G_s.$$

Example $G = GL_n(\mathbb{R})$

$$S = M_n(\mathbb{R})$$

$$G \curvearrowright S \text{ as } g \cdot m = gm^t g$$

Claim: This is an action

Pf: • $1 \cdot s = 1 \cdot m \cdot 1 = 1 \cdot m \cdot 1 = m$

$$\begin{aligned} \bullet g_1 \cdot (g_2 \cdot m) &= g_1 \cdot (g_2 m^t g_2) = g_1 (g_2 m^t g_2)^t g_1 \\ &= (g_1 g_2) m (g_1 g_2)^t = (g_1 g_2) \cdot m. \end{aligned}$$

$\Rightarrow$ If A is fixed, then

$$g \in G \text{ such that } g \cdot A = A$$

if A is fixed, then

$$G_A = \{g \in GL_n(\mathbb{R}) : gA^t g = A\} \subseteq GL_n(\mathbb{R})$$

a subsp.

Ex $S_n \curvearrowright \{1, 2, \dots, n\}$. The stabilizer of n is isomorphic to S_{n-1} (permutations of $\{1, 2, \dots, n-1\}$)

Propⁿ $G \curvearrowright S$, $s \in S$, $H = G_s$ is the stabilizer of the elt s .

a) If $a, b \in G$ then $as = bs \iff a^{-1}b \in H \iff b \in aH$

b) Suppose $s' = as$, and let H' denote the stabilizer of s' .

$$\text{Then } H' = aHa^{-1} = \{g \in G : g = aha^{-1} \text{ for some } h \in H\}.$$

Pf: a) $as = bs \iff s = a^{-1}bs \xrightarrow[\text{applying } m_{a^{-1}}]{} a^{-1}b \in H \iff b \in aH.$

b) $\underline{aHa^{-1} \subseteq H'}$: Suppose $h \in H$. Then

$$(aha^{-1}) \cdot (s') = (aha^{-1}) \cdot (as) = ah(s) = as = s'$$

• $\underline{H' \subseteq aHa^{-1}}$: Suppose $g' \in H'$, then $g' = ag a^{-1}$ for some $g \in G$ (Pf: $g = a^{-1}g'a$)

$$g \in C \quad (\text{Pf: } g = \bar{a}^{-1} g' a)$$

$$as = s' = g's' = (aga^{-1})s' = (aga^{-1})(as) = a(gs)$$

$$\xrightarrow{m_{\bar{a}^{-1}}} s = gs \Rightarrow g \in H \Leftrightarrow g' \in aH\bar{a}^{-1} \checkmark$$

Recall G gp, $H \subseteq G$ is a subgp, then

$$G \curvearrowright \underbrace{G/H}$$

the set of left H cosets.

Notation: if $C = aH$ is a left H coset, write $[C]$ for the associated elt of G/H

$$\text{Action: } G \curvearrowright G/H \text{ is } g \cdot [C] = [gC] = [\{gc : c \in C\}]$$

Propⁿ $H \subseteq G$ is a subgp.

- a) The action $G \curvearrowright G/H$ is transitive
- b) The stabilizer of $[H]$ is H .

Pf: a) Recall that transitive means there is one G orbit.

And indeed, if $[C] \in G/H$, $[C] = [aH] = a \cdot [H]$.

$$\text{b) } g \cdot [H] = [H]? \quad g \cdot [H] = [gH] = [H] \\ \Leftrightarrow g \in H.$$

✓

Propⁿ S a set, $G \curvearrowright S$, $s \in S$, $\mathcal{O}_s$ is the orbit of s ,
 and $H = G_s$ is the stabilizer of s . Then there exists
 a bijection $\Sigma: G/H \rightarrow \mathcal{O}_s$ defined as

$$\Sigma([aH]) = a \cdot s.$$

Moreover, Σ is compatible with the group actions on both
 sides

$$\Sigma(g \cdot [C]) = g \cdot \Sigma([C]).$$

Pf: Our defⁿ of Σ $\uparrow$ depends on representative a .
 a priori

$$C = aH = bH \text{ then } \begin{matrix} \Sigma([C]) = as \\ \Sigma([C]) = bs \end{matrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} \text{ are they equal?}$$

$$aH = bH \implies b = ah \text{ for some } h \in H$$

$$\implies bs = (ah)s = as \text{ because } H \text{ stabilizes } s.$$

$$\begin{aligned} \Sigma(g \cdot [C]) &= \Sigma(g \cdot [aH]) = \Sigma([gaH]) \\ &= (ga) \cdot s \\ &= g(as) = g \cdot \Sigma([C]) \end{aligned}$$
